Grading guide, Pricing Financial Assets, June 2013

1. Let the price of a traded stock, S, paying no dividends be modelled by the geometric Brownian motion

$$dS = \mu S dt + \sigma S dz$$

where μ and σ are constants, and where dt and dz are the standard short hand notations for a small time-step and a Brownian increment.

Consider a forward contract on the stock with a time to maturity of T. Let the current ($t = 0$) price of the stock be S_0, and assume that the risk-free rate of interest is a constant r.

(a) What is relationship between the forward price F_0 at $t = 0$ and the stock price? Comment on the result.

(b) Use Ito's lemma derive the process followed by the forward price F.

(c) Compare the price of a European call option on the stock with the price of a European call option on the forward when the two options have the same strikes and maturities.

Solution:

(a) An arbitrage argument should be used to show

$$F_0 = S_0 \mathrm{e}^{rT}$$

(b) Use Ito's lemma to get

$$dF = (\mu - r)F dt + \sigma F dz$$

cf. Hull p. 292. It can be noted that the drift of the forward is lower than that of the stock, and that the forward price in the risk-neutral world will be a martingale.

(c) Comparing the processes for F and S it can be noted that the process for F corresponds to the process for S if the stock paid a constant dividend yield of r, cf. Hull p. 370.

The Black-Scholes-Merton formula for pricing a European call on a stock paying a constant dividend yield can thus be reinterpreted to cover the case of the underlying being a forward. The economic interpretation is that the forward has a postponement of payment (a carry advantage compared to buying and holding the stock) but receives no dividends (a carry disadvantage in the same comparison). Other formulations of the relative pricing will also be valid.

2. (a) What is a ratings transitions matrix? What is the typical structure af such a matrix?

(b) What assumptions are needed to use the matrix to derive an estimate of the likelihood of a default of a rated issuer within a certain time horizon? And will this estimate be a real world or a risk neutral probability?

(c) Suppose the ratings transitions matrix is used to determine the default probability of two rated issuers. Describe in general terms how a Copula function can be used to assess the probability of a default of both issuers within a certain time horizon.

Solution:

(a) The definition is in Hull p 541-2. The structure can be brought to be of the form of a finite state transition probability matrix with an absorbing state (default). Typically the largest entries are on the diagonal with more dispersion for the lower quality ratings.

(b) We get real world probabilities, not risk adjusted. The assumptions that we need are

- that the transition probabilities are constant and equal to the historical frequencies used
- that issuers going from rated to non-rated can be assumed to be distributed in a pre-described fashion e.g. proportional to the other issuers' transitions to actual ratings.

(c) The rating transition matrix can give the marginal probabilities of default for the issuers. The Copula can describe the interdependence of defaults using the marginals as inputs e.g. using the Gaussian Copula, cf. Hull p. 538 and a very simple numerical example using a ratings transition matrix on p. 542.

3. Consider an interest rate cap with a life of T, a principal of L, and a cap rate of R_K. Consider reset dates $0 = t_0 < t_1 < t_2, \ldots < t_n$, and let R_k be the Libor rate for the period from t_k to t_{k+1} known at t_k.

(a) Describe the payments of the cap, and define a caplet.

(b) Show that the cap can be considered as a portfolio of European put options on zero-coupon bonds.

(c) A standard market practice is to price a caplet (at $t = 0$) with the Black formula:

$$Caplet^{Black}(0, k, L, R_K, \sigma_k) = LP(0, t_{k+1})\tau_k(F_k N(d_1) - R_K N(d_2))$$

with

$$d_1 = \frac{ln(F_k/R_K) + 0.5\sigma_k^2 t_k}{\sigma_k\sqrt{t_k}}$$
$$d_2 = d_1 - \sigma_k\sqrt{t_k}$$

where $P(t, T)$ is the price of a zero coupon bond at t maturing at T, F_k is the forward rate at 0 for the time interval (t_k, t_{k+1}) with length τ_k, and σ_k the volatility of this rate. What assumptions can justify this formula?

Solution:

(a)

Definition 0.1 (Caps). *Let the Libor-rate set at t_k for the period (t_k, t_{k+1}) be R_k. Consider a maturity T, a principal of L and a cap rate of R_K. Let reset dates $t_1, t_2, \ldots, t_n$ be given and $t_{n+1} = T$. A Cap is a contract that gives its holder the payments*

$$L\tau_k max(R_k - R_K; 0)$$

at time t_{k+1} (i.e. in arrears), where τ_t is the time fraction between t_k and t_{k+1}

Hull defines a cap with payments in arrears p.653. The potential payment at each payoff date defines the payment of a single caplet starting on the previous reset date , cf. Hull p. 653.

(b) Cf. Hull p. 654.

- A cap can equivalently be seen as a portfolio of puts on properly defined Zero Coupon Bonds (ZCBs)

$$
\begin{aligned}
PV_{t_k}(L\tau_k max(R_k - R_K; 0)) = \\
\frac{L\tau_k}{1 + R_k\tau_k} max(R_k - R_K; 0) = \\
\frac{L\tau_k}{1 + R_k\tau_k} max((1/\tau_k + R_k) - (1/\tau_k + R_K); 0) = \\
max(L - L\frac{1 + R_K\tau_k}{1 + R_k\tau_k}; 0)
\end{aligned}
$$

- The second part in the first argument under the max-operator is the value at t_k of a ZCB with payoff $L(1 + R_K\tau_k)$ at t_k; thus the payoff is a put on this ZCB with strike L

(c) Cf. Hull p. 657. The application of the Black model is consistent with a world that is forward risk neutral wrt a ZCB maturing at t_{k+1}. Under this probability measure assume R_k has a lognormal distribution with a standard deviation of $\ln(R_k)$ of $\sigma_k\sqrt{t_k}$. We also have under this measure that the expectation of R_k is the forward rate F_k.